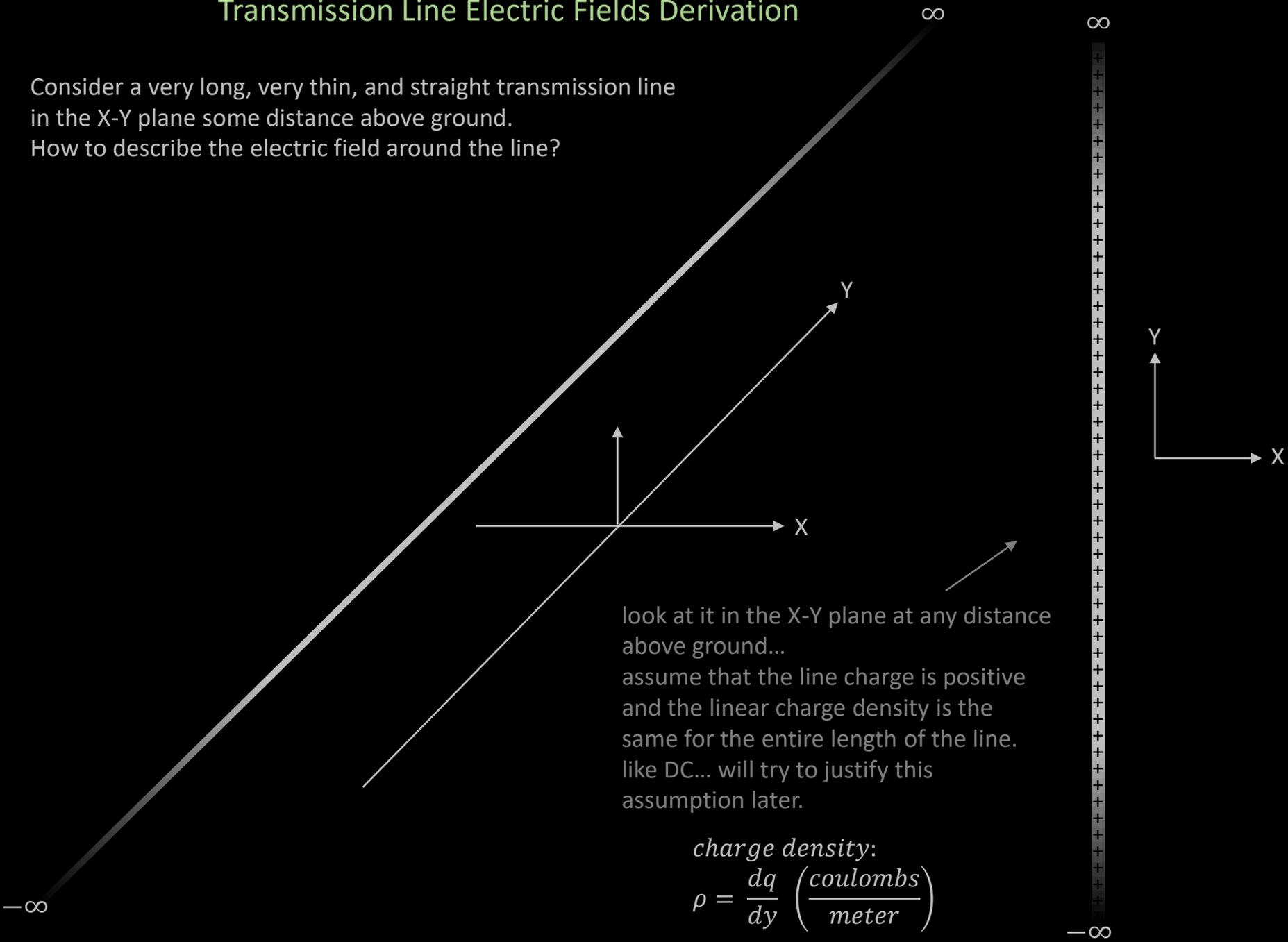


# Transmission Line Electric Fields Derivation

Consider a very long, very thin, and straight transmission line in the X-Y plane some distance above ground. How to describe the electric field around the line?



look at it in the X-Y plane at any distance above ground...  
assume that the line charge is positive and the linear charge density is the same for the entire length of the line.  
like DC... will try to justify this assumption later.

charge density:

$$\rho = \frac{dq}{dy} \left( \frac{\text{coulombs}}{\text{meter}} \right)$$

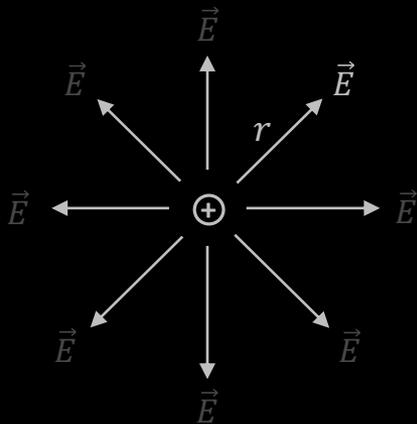
## Transmission Line Electric Fields (cont.)

Show two (2) methods for finding the electric field some distance from the center of the conductor

Method one (1): Coulomb's Law

the electric field from a point charge is proportional to the charge and inversely proportional to the distance from the charge squared.

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}$$



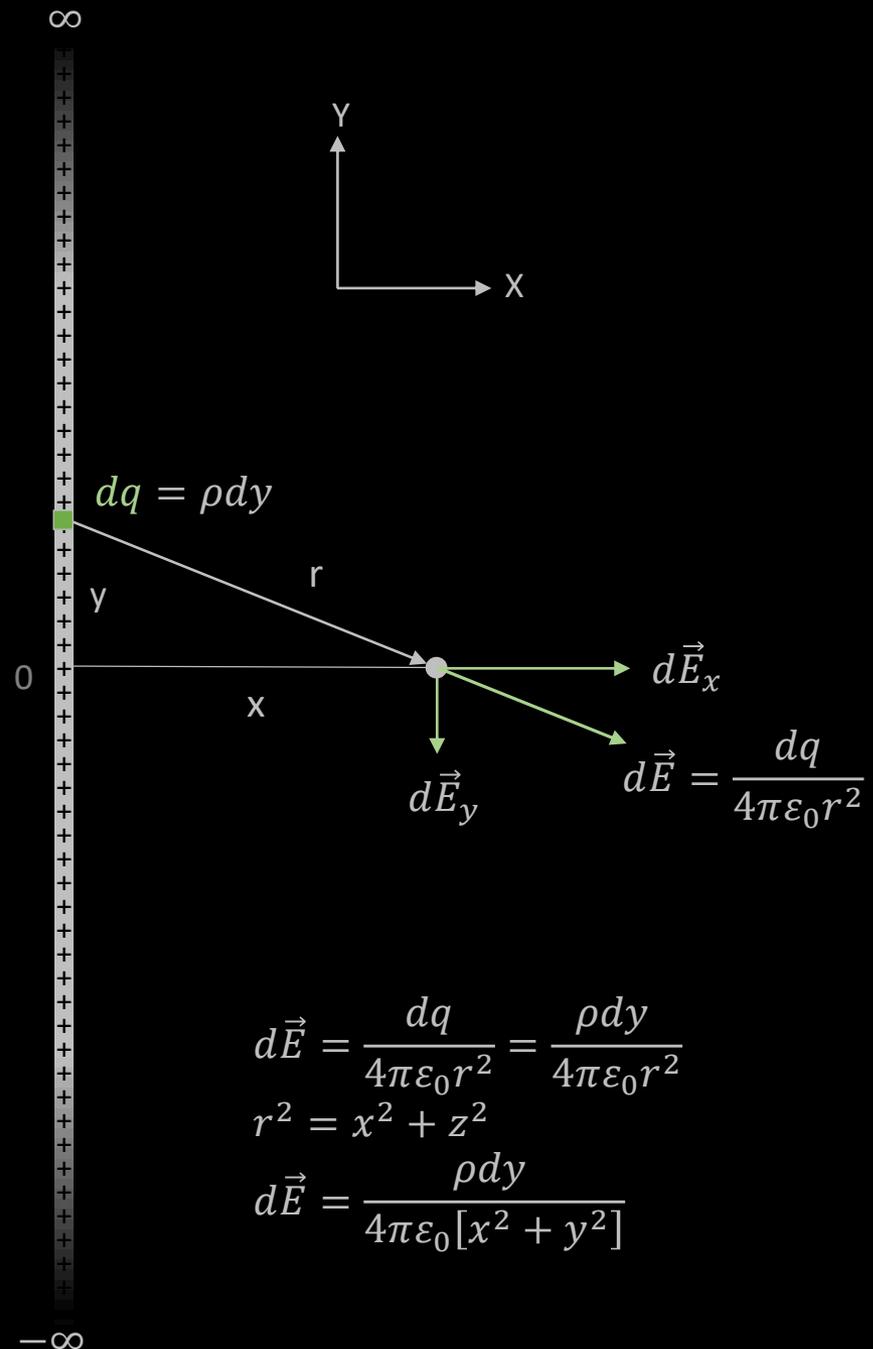
the electric field is radial from the point charge in all directions

permeability of free space

$$\mu_0 = 4\pi \times 10^{-7} \frac{H}{m}$$

permittivity of free space

$$\epsilon_0 = 8.854 \times 10^{-12} \frac{F}{m}$$



# Transmission Line Electric Fields (cont.)

Method one (1): Coulomb's Law

get the horizontal and vertical components of the infinitesimally small electric field contribution

$$d\vec{E}_x = d\vec{E} \cos \alpha$$

$$\cos \alpha = \frac{x}{r}$$

$$d\vec{E}_x = \frac{\rho dz}{4\pi\epsilon_0[x^2 + y^2]} \frac{x}{\sqrt{x^2 + y^2}}$$

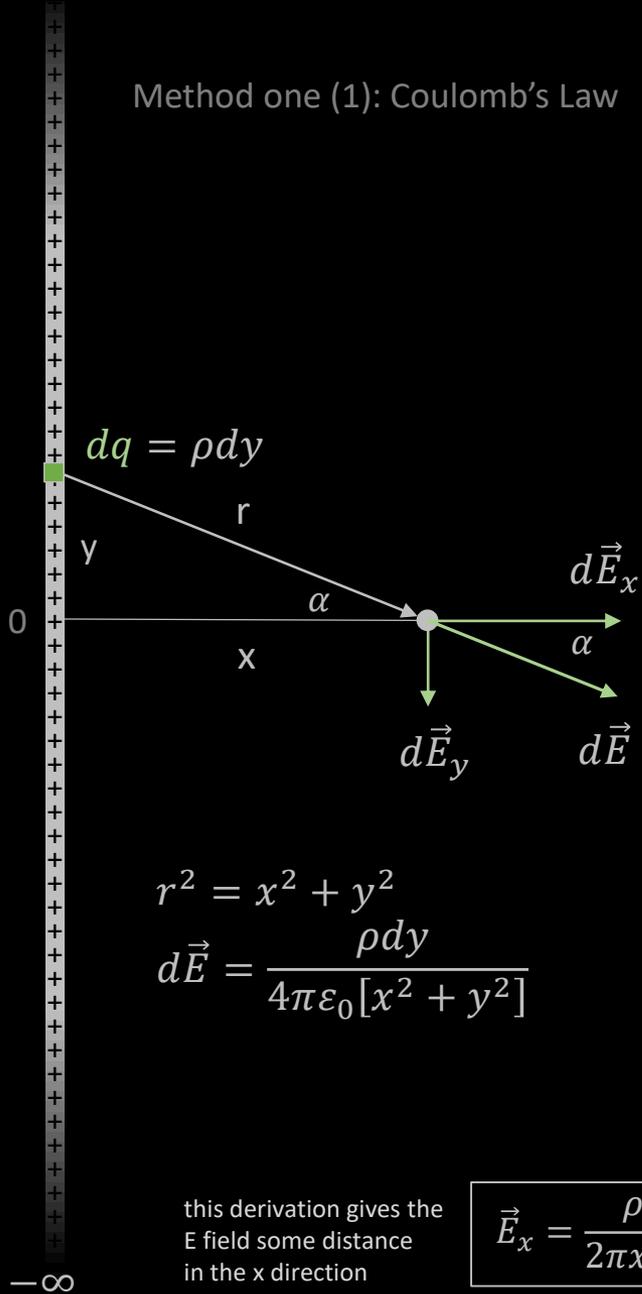
$$d\vec{E}_x = \frac{\rho x dy}{4\pi\epsilon_0[x^2 + y^2]^{\frac{3}{2}}}$$

$$d\vec{E}_y = d\vec{E} \sin \alpha$$

$$\sin \alpha = \frac{y}{r}$$

$$d\vec{E}_y = \frac{\rho dy}{4\pi\epsilon_0[x^2 + y^2]} \frac{y}{\sqrt{x^2 + y^2}}$$

$$d\vec{E}_y = \frac{\rho y dy}{4\pi\epsilon_0[x^2 + y^2]^{\frac{3}{2}}}$$



sum up the contributions to get total electric field

$$\vec{E}_x = \frac{\rho x}{4\pi\epsilon_0} \int_{-\infty}^{\infty} [x^2 + y^2]^{-\frac{3}{2}} dy$$

$$\vec{E}_x = \frac{\rho x}{4\pi\epsilon_0} \left[ \frac{y}{x^2 \sqrt{x^2 + y^2}} \right]_{-\infty}^{\infty}$$

$$\vec{E}_x = \frac{\rho}{4\pi x \epsilon_0} \left[ \frac{y}{\sqrt{x^2 + y^2}} \right]_{-\infty}^{\infty}$$

$$\vec{E}_x = \frac{\rho}{4\pi x \epsilon_0} \left[ \frac{\infty}{\sqrt{x^2 + \infty^2}} - \frac{-\infty}{\sqrt{x^2 + \infty^2}} \right]$$

$$\vec{E}_x = \frac{\rho}{4\pi x \epsilon_0} [1 + 1] = \frac{\rho}{2\pi x \epsilon_0}$$

$$\vec{E}_y = \frac{\rho}{4\pi\epsilon_0} \int_{-\infty}^{\infty} y [x^2 + y^2]^{-\frac{3}{2}} dy$$

$$\vec{E}_y = \frac{-\rho}{4\pi\epsilon_0} \left[ \frac{1}{\sqrt{x^2 + y^2}} \right]_{-\infty}^{\infty}$$

$\vec{E}_y = 0$  (symmetry shows it)

$$r^2 = x^2 + y^2$$

$$d\vec{E} = \frac{\rho dy}{4\pi\epsilon_0[x^2 + y^2]}$$

we can change notation and say that the E field is inversely proportional to some radial distance "r" from the conductor

this derivation gives the E field some distance in the x direction

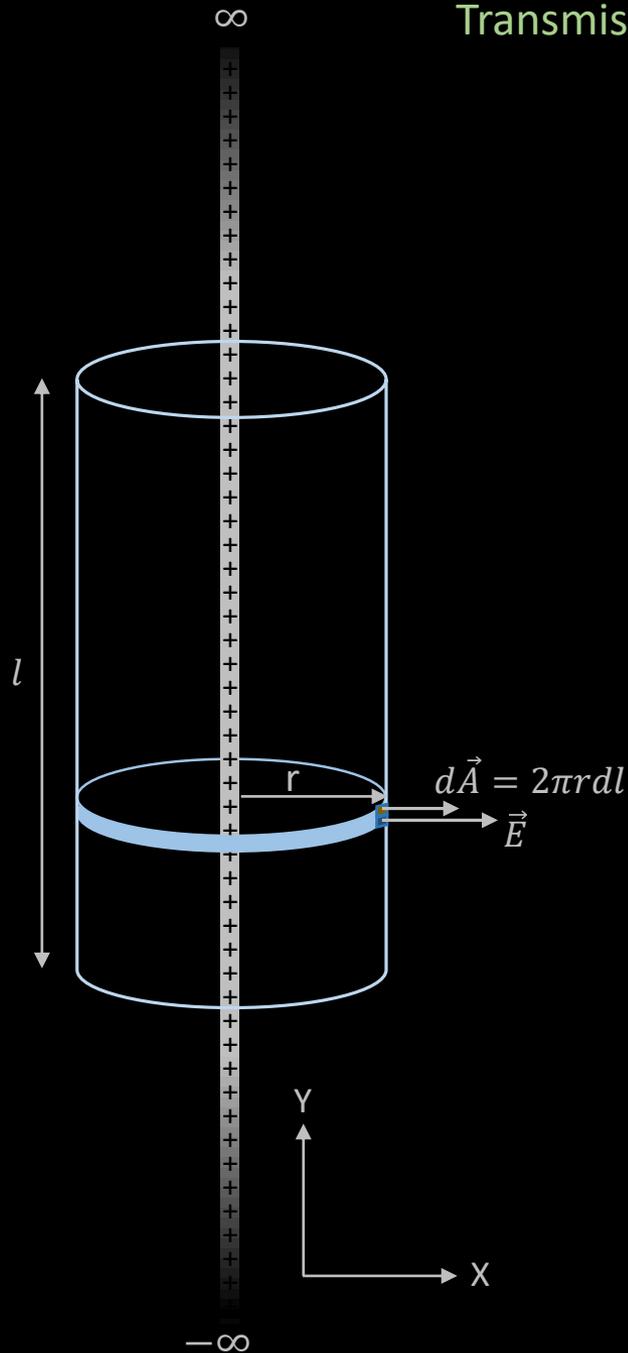
$$\vec{E}_x = \frac{\rho}{2\pi x \epsilon_0}$$

had we considered the E field in the z direction... would yield same result

$$\vec{E}_z = \frac{\rho}{2\pi z \epsilon_0}$$

$$\vec{E} = \frac{\rho}{2\pi r \epsilon_0} \hat{r}$$

## Transmission Line Electric Fields (cont.)



### Method two (2): Gauss's Law

the electric flux through any closed surface is proportional the total charge inside  
(only works if charge density is constant)

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{inside}}{\epsilon_0}$$

total charge on this length of line is:

$$Q_{inside} = \rho \int_0^l dl = \rho l$$

since  $\vec{E}$  is parallel to  $d\vec{A}$  everywhere on the surface:

$$\vec{E} \cdot d\vec{A} = E da$$

$$\int_0^l E da = 2\pi r E \int_0^l dl = 2\pi r E l$$

$\therefore$

$$2\pi r E l = \frac{\rho l}{\epsilon_0}$$

$$E = \frac{\rho}{2\pi r \epsilon_0}$$

$$\vec{E} = \frac{\rho}{2\pi r \epsilon_0} \hat{r}$$

← much easier than using Coulombs's Law

## Transmission Line Electric Fields (cont.)

try to justify why using an infinitely long line is ok

Go back to Coulomb's Law where we found  
the total electric field for an infinitely long line:

$$\vec{E}_x = \frac{\rho}{4\pi x \epsilon_0} \left[ \frac{y}{\sqrt{x^2 + y^2}} \right]_{-\infty}^{\infty}$$

and consider a line of finite length =  $l$

$$\vec{E}_x = \frac{\rho}{4\pi x \epsilon_0} \left[ \frac{y}{\sqrt{x^2 + y^2}} \right]_{-\frac{l}{2}}^{\frac{l}{2}} = \frac{\rho}{2\pi x \epsilon_0} \left[ \frac{y}{\sqrt{x^2 + y^2}} \right]_0^{\frac{l}{2}}$$

$$\vec{E}_x = \frac{\rho}{2\pi x \epsilon_0} \left[ \frac{\frac{l}{2}}{\sqrt{x^2 + \left[\frac{l}{2}\right]^2}} \right]$$

If half the line length is much greater than the distance  
of the point of interest away from the line:

$$\text{if: } \frac{l}{2} > 10x$$

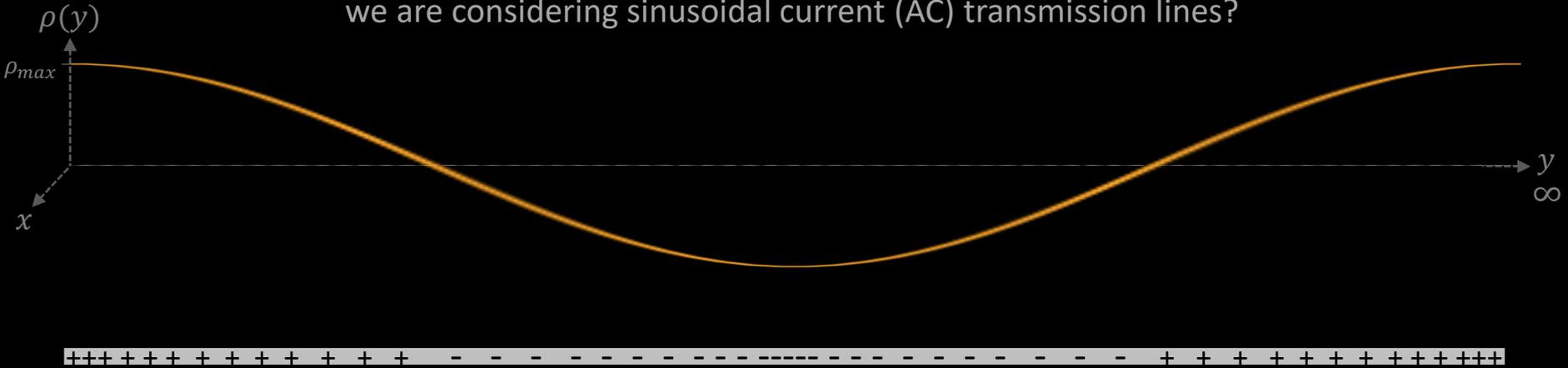
$$\vec{E}_x \approx \frac{\rho}{2\pi x \epsilon_0} \left[ \frac{\frac{l}{2}}{\sqrt{\left[\frac{l}{2}\right]^2}} \right] = \frac{\rho}{2\pi x \epsilon_0}$$

$$\vec{E}_x \approx \frac{\rho}{2\pi x \epsilon_0}$$

very close to the same result, and this is easily satisfied.  
only interested in the fields within 20m of the line or so...  
and transmission lines will always be greater than 400m.

## Transmission Line Electric Fields (cont.)

is using a constant charge density ok when we are considering sinusoidal current (AC) transmission lines?



Assumptions:

- the line is sourced from the left and the load is some distance down the line to the right
- operating frequency is  $f = 60 \text{ Hz}$  (period =  $T = 16.67 \text{ ms}$ )
- conductor is aluminum with permeability =  $\mu_0$  permittivity =  $\epsilon_0$

the electromagnetic energy wave will propagate down the line at a velocity =  $\frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$  (speed of light)

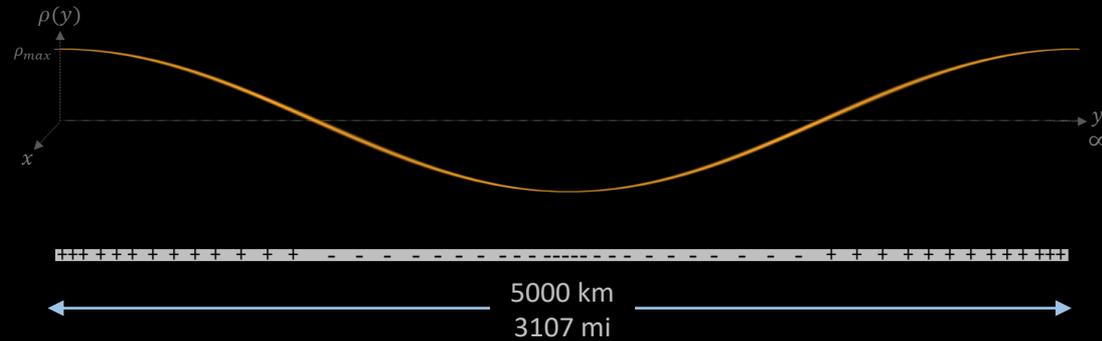
How far does the wave travel in one period?

*velocity \* time = distance*

$$y = cT = \frac{c}{f} = \frac{3 * 10^8}{60} = 5000 \text{ km (3107 mi)}$$

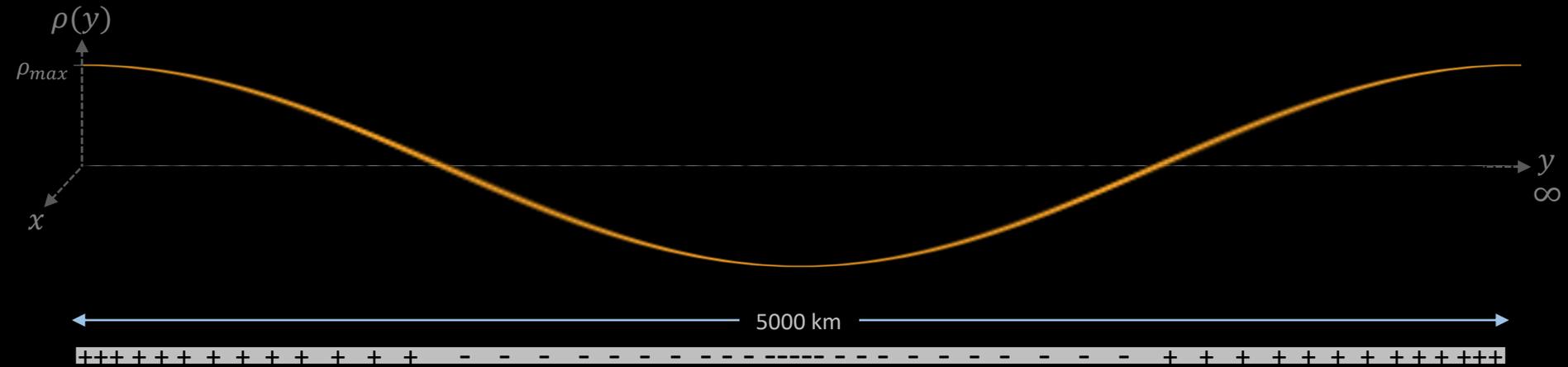
*this is the wavelength =  $\lambda$*

$$\lambda = \frac{c}{f}$$



# Transmission Line Electric Fields (cont.)

## evaluating constant charge density assumption



Say we are interested the E fields less than some  $r$  distance away from the line.

Another way to state the infinite line assumption is that only about  $\pm 10r$  length of the line at that point has significant contribution to the E fields.

we are only interested in the E fields within about 20m or so from a three phase line configuration.

So we have to ask...

Could the line charge density of  $\pm 200m$  of line be considered constant?

Probably a better way to ask it...

How much does the charge density change over 400m of line length?

What we are actually wanting to know is...

what portion of the wavelength is 400m?

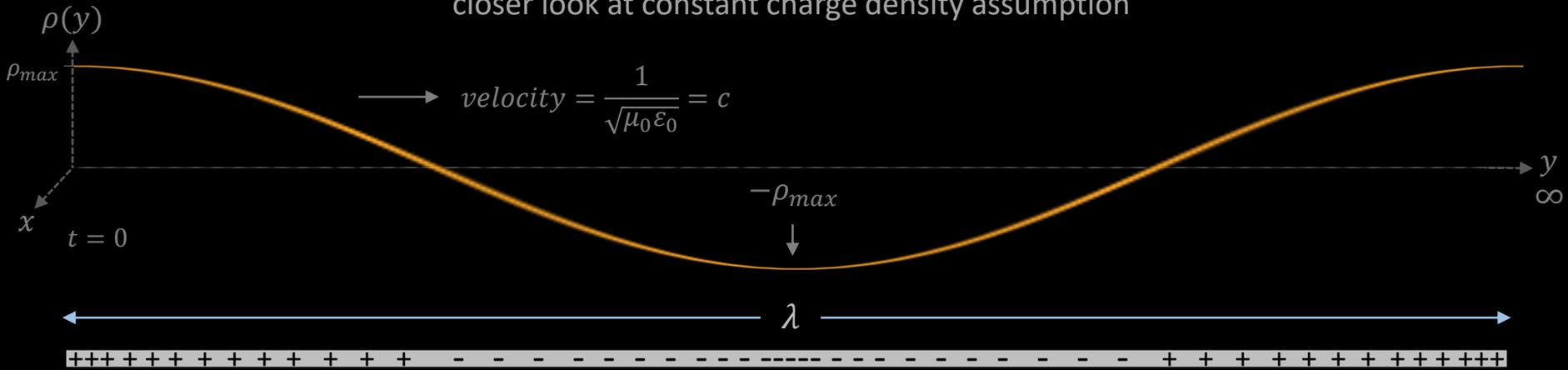
$$\frac{400m}{5000km} = \frac{\lambda}{12000}$$

It seems reasonable to assume that the charge density does not change much over 0.008% of the wavelength

But let's look a little closer

## Transmission Line Electric Fields (cont.)

closer look at constant charge density assumption



let the magnitude of the maximum charge density =  $\rho_{max}$

the charge density as a function of distance can be written as:  $\rho(y) = \rho_{max} \cos \left[ \frac{2\pi}{\lambda} y \right]$

we know the charge on the line is related to the instantaneous current because the definition of current is:  $i = \frac{dQ}{dt}$

we know the velocity of propagation is:  $\frac{dy}{dt} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$

so:  $\frac{dQ}{dy} = \frac{dQ}{dt} \frac{dt}{dy} = i \sqrt{\mu_0 \epsilon_0} \quad \Leftarrow$  this is the instantaneous charge density at  $y = 0$  (at some instant of time)

where:  $i = I \cos(\omega t) \quad \Leftarrow$  instantaneous RMS current

so:  $\rho(t) = \underbrace{\sqrt{\mu_0 \epsilon_0} I}_{\text{this is } \rho_{max}} \cos(\omega t) \quad \therefore \quad \rho(y) = \sqrt{\mu_0 \epsilon_0} I \cos \left[ \frac{2\pi}{\lambda} y \right]$

this is  $\rho_{max}$

we can write the line charge as a function of time and distance by adding a phase shift:

$$\rho(t, y) = \sqrt{\mu_0 \epsilon_0} I \cos \left[ \frac{2\pi}{\lambda} y - \omega t \right]$$

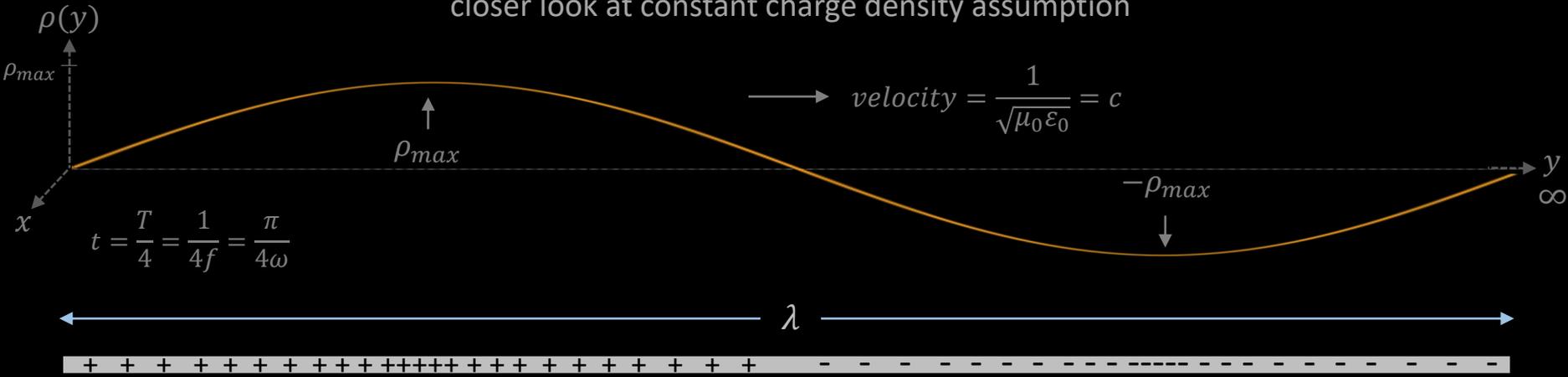
the figure above is a “snapshot” in time at  $\omega t = 0$

charge density is max at  $y = 0, \frac{\lambda}{2}, \lambda$

charge density is zero at  $y = \frac{\lambda}{4}, \frac{3\lambda}{4}$

# Transmission Line Electric Fields (cont.)

closer look at constant charge density assumption



Now a "snapshot" in time when  $\omega t = \frac{\pi}{4}$

$$\rho\left(\frac{T}{4}, y\right) = \sqrt{\mu_0 \epsilon_0} I \cos\left[\frac{2\pi}{\lambda} y - \omega t\right]$$

$$\rho(t, y) = \sqrt{\mu_0 \epsilon_0} I \cos\left[\frac{2\pi}{\lambda} y - \omega t\right]$$

charge density is max at  $y = \frac{\lambda}{4}, \frac{3\lambda}{4}$

charge density is zero at  $y = 0, \frac{\lambda}{2}, \lambda$

This is how we will evaluate the constant charge density assumption...  
 Re-derive the E fields with Coulomb's Law using a charge density that changes over the length of the line (for any "snapshot" in time).

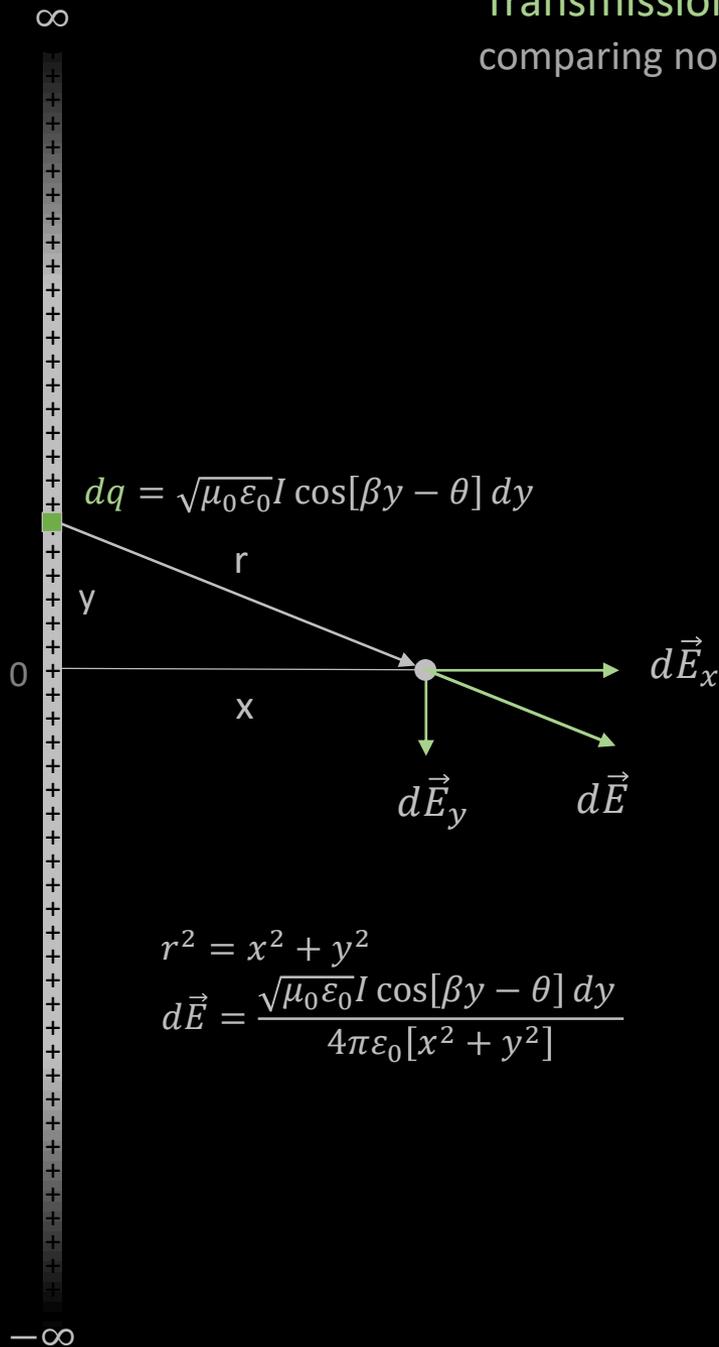
$$\rho = \frac{dQ}{dy} = \sqrt{\mu_0 \epsilon_0} I \cos[\beta y - \theta]$$

where:  $\beta = \frac{2\pi}{\lambda}$     $\theta = \omega t$

we are most interested in when and where the current or charge density crosses zero...  
 (this is where the charge density has maximum change)  
 keep it as simple as possible and use E field point of interest at *position* = 0

# Transmission Line Electric Fields (cont.)

comparing non-linear AC charge distribution



$$dq = \sqrt{\mu_0 \epsilon_0} I \cos[\beta y - \theta] dy$$

$$r^2 = x^2 + y^2$$

$$d\vec{E} = \frac{\sqrt{\mu_0 \epsilon_0} I \cos[\beta y - \theta] dy}{4\pi \epsilon_0 [x^2 + y^2]}$$

$$d\vec{E}_x = d\vec{E} \cos \alpha$$

$$\cos \alpha = \frac{x}{r}$$

$$d\vec{E}_x = \frac{\sqrt{\mu_0 \epsilon_0} I \cos[\beta y - \theta]}{4\pi \epsilon_0 [x^2 + y^2]} \frac{x dy}{\sqrt{x^2 + y^2}}$$

$$d\vec{E}_x = \frac{\sqrt{\mu_0 \epsilon_0} I x \cos[\beta y - \theta] dy}{4\pi \epsilon_0 [x^2 + y^2]^{\frac{3}{2}}}$$

trig identity:

$$\cos[\beta y - \theta] = \cos \beta y \cos \theta + \sin \beta y \sin \theta$$

since  $\beta y$  is a very small angle ...

use small angle approximation:

$$\sin[\beta y] \approx \beta y \quad \cos[\beta y] \approx 1$$

$$\sin[\theta - \beta y] \approx \cos \theta + \beta y \sin \theta$$

$$d\vec{E}_x = \frac{\sqrt{\mu_0 \epsilon_0} I x [\cos \theta + \beta y \sin \theta] dy}{4\pi \epsilon_0 [x^2 + y^2]^{\frac{3}{2}}}$$

$$d\vec{E}_x = \frac{\sqrt{\mu_0 \epsilon_0} I x [\cos \theta + \beta y \sin \theta] dy}{4\pi \epsilon_0 [x^2 + y^2]^{\frac{3}{2}}}$$

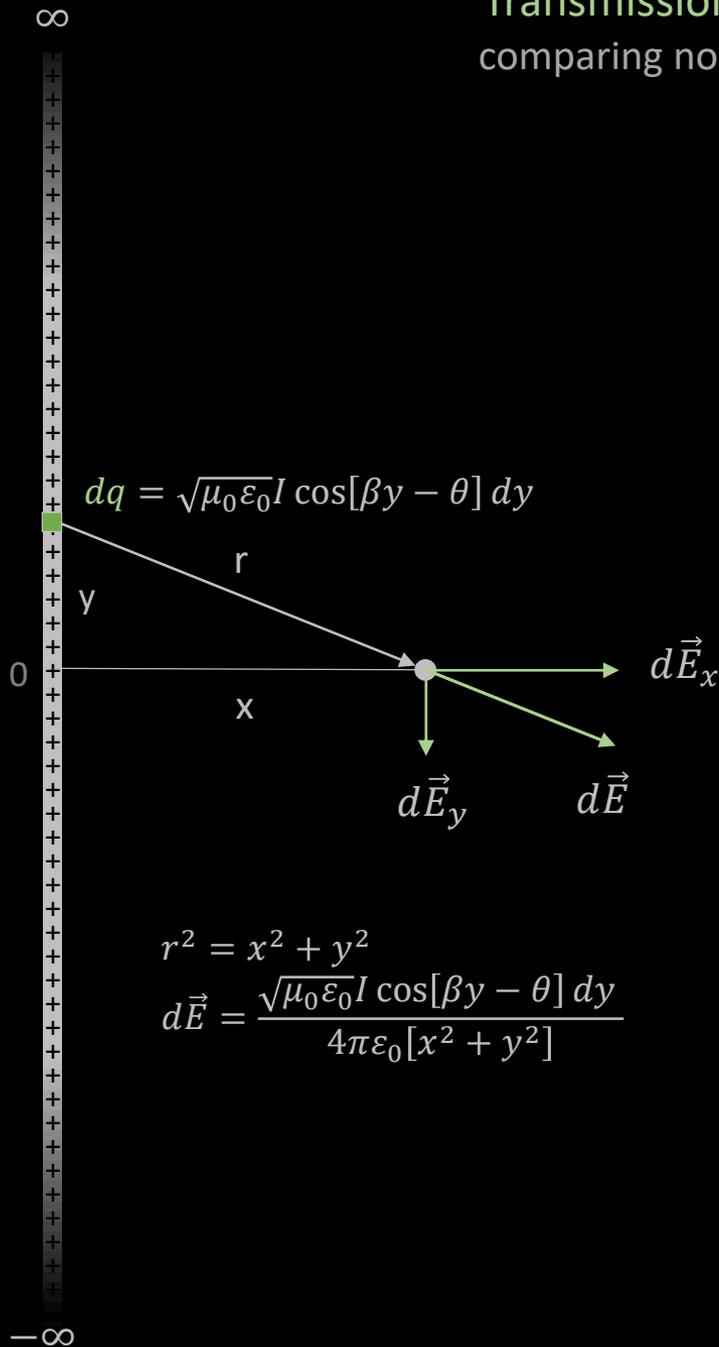
$$d\vec{E}_x = \frac{\sqrt{\mu_0 \epsilon_0} I}{4\pi \epsilon_0} \left[ \frac{x \cos \theta dy}{[x^2 + y^2]^{\frac{3}{2}}} + \frac{x \beta \sin \theta y dy}{[x^2 + y^2]^{\frac{3}{2}}} \right]$$

sum up the contributions to get total electric field in x direction

$$\vec{E}_x = \frac{\sqrt{\mu_0 \epsilon_0} I}{4\pi \epsilon_0} \left[ \int_{-\infty}^{\infty} \frac{x \cos \theta dy}{[x^2 + y^2]^{\frac{3}{2}}} + \int_{-\infty}^{\infty} \frac{x \beta \sin \theta y dy}{[x^2 + y^2]^{\frac{3}{2}}} \right]$$

# Transmission Line Electric Fields (cont.)

comparing non-linear AC charge distribution



$$dq = \sqrt{\mu_0 \epsilon_0} I \cos[\beta y - \theta] dy$$

$$r^2 = x^2 + y^2$$

$$d\vec{E} = \frac{\sqrt{\mu_0 \epsilon_0} I \cos[\beta y - \theta] dy}{4\pi \epsilon_0 [x^2 + y^2]}$$

$$d\vec{E}_y = d\vec{E} \sin \alpha$$

$$\sin \alpha = \frac{y}{r}$$

$$d\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I \cos[\beta y - \theta]}{4\pi \epsilon_0 [x^2 + y^2]} \frac{y dy}{\sqrt{x^2 + y^2}}$$

$$d\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I \cos[\beta y - \theta] y dy}{4\pi \epsilon_0 [x^2 + y^2]^{\frac{3}{2}}}$$

trig identity:

$$\cos[\beta y - \theta] = \cos \beta y \cos \theta + \sin \beta y \sin \theta$$

since  $\beta y$  is a very small angle ...

use small angle approximation:

$$\sin[\beta y] \approx \beta y \quad \cos[\beta y] \approx 1$$

$$\sin[\theta - \beta y] \approx \cos \theta + \beta y \sin \theta$$

$$d\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I [\cos \theta + \beta y \sin \theta] y dy}{4\pi \epsilon_0 [x^2 + y^2]^{\frac{3}{2}}}$$

$$d\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I [\cos \theta + \beta y \sin \theta] y dy}{4\pi \epsilon_0 [x^2 + y^2]^{\frac{3}{2}}}$$

$$d\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I}{4\pi \epsilon_0} \left[ \frac{\cos \theta y dy}{[x^2 + y^2]^{\frac{3}{2}}} + \frac{\beta \sin \theta y^2 dy}{[x^2 + y^2]^{\frac{3}{2}}} \right]$$

sum up the contributions to get total electric field in y direction

$$\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I}{4\pi \epsilon_0} \left[ \int_{-\infty}^{\infty} \frac{\cos \theta y dy}{[x^2 + y^2]^{\frac{3}{2}}} + \int_{-\infty}^{\infty} \frac{\beta \sin \theta y^2 dy}{[x^2 + y^2]^{\frac{3}{2}}} \right]$$

## Transmission Line Electric Fields (cont.)

comparing non-linear AC charge distribution

$$\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I}{4\pi \epsilon_0} \left[ \int_{-\infty}^{\infty} \frac{\cos \theta y dy}{[x^2 + y^2]^{\frac{3}{2}}} + \int_{-\infty}^{\infty} \frac{\beta \sin \theta y^2 dy}{[x^2 + y^2]^{\frac{3}{2}}} \right]$$

$$\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I}{4\pi \epsilon_0} \left[ \cos \theta \left[ \frac{-1}{\sqrt{x^2 + y^2}} \right]_{-\infty}^{\infty} + \beta \sin \theta \left[ \sinh^{-1} \left[ \frac{y}{x} \right] - \frac{y}{\sqrt{x^2 + y^2}} \right]_{-\infty}^{\infty} \right]$$

$$\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I}{4\pi \epsilon_0} \left[ \cos \theta \left[ \frac{-1}{\sqrt{x^2 + y^2}} \right]_{-\infty}^{\infty} + \beta \sin \theta \left[ \left[ \sinh^{-1} \left[ \frac{y}{x} \right] \right]_{-\infty}^{\infty} - \left[ \frac{y}{\sqrt{x^2 + y^2}} \right]_{-\infty}^{\infty} \right] \right]$$

$$\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I}{4\pi \epsilon_0} \left[ \cos \theta \left[ \frac{-1}{\sqrt{x^2 + \infty^2}} - \frac{-1}{\sqrt{x^2 + \infty^2}} \right] + \beta \sin \theta \left[ \sinh^{-1} \left[ \frac{\infty}{x} \right] - \sinh^{-1} \left[ \frac{-\infty}{x} \right] - \left[ \frac{\infty}{\sqrt{x^2 + \infty^2}} - \frac{-\infty}{\sqrt{x^2 + \infty^2}} \right] \right] \right]$$

$$\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I}{4\pi \epsilon_0} \left[ [0 - 0] + \beta \sin \theta \left[ 2 \sinh^{-1} \left[ \frac{\infty}{x} \right] - 2 \right] \right]$$

$$\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I}{4\pi \epsilon_0} \left[ 2\beta \sin \theta \left[ \sinh^{-1} \left[ \frac{\infty}{x} \right] - 1 \right] \right]$$

$$\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I}{2\pi \epsilon_0} \underbrace{\beta \sin \theta \left[ \sinh^{-1} \left[ \frac{\infty}{x} \right] - 1 \right]}_{\text{Indeterminate y component!}}$$

Indeterminate y component!

But there is a y (non-radial) component... maximum at zero crossing as expected

Question is...

Is it large enough to be significant?

## Transmission Line Electric Fields (cont.)

comparing non-linear AC charge distribution

summary so far

$$\vec{E}_x = \frac{\sqrt{\mu_0 \epsilon_0} I \cos \theta}{2\pi \epsilon_0 x}$$

The E field in the x or radial direction from the line is the same as when a constant charge distribution was assumed.

This makes sense because the charge is seen as increasing (or decreasing) at the same rate from the position of interest.

In other words...

the charge (density) gets less as you look down the line,

the charge (density) gets greater as you look up the line.

So the increased E field contribution from up the line,

is offset by the decreased contribution from down line.

Yielding the same result as if you assumed a constant line charge.

$$|\vec{E}_x| = \frac{\sqrt{\mu_0 \epsilon_0} I}{2\pi \epsilon_0 x}$$

$$\vec{E}_y = \frac{\sqrt{\mu_0 \epsilon_0} I}{2\pi \epsilon_0} \beta \sin \theta \left[ \sinh^{-1} \left[ \frac{\infty}{x} \right] - 1 \right]$$

The E field in the y or parallel direction of the line is not zero as when a constant charge distribution was assumed.

Question is...

Is it large enough to be significant?

Let's integrate the charge contributions over a very long but finite line length.

Be conservative and say we are interested in E fields within 50m of the line.

This means that up to  $\pm 500$ m of the line charge could have significant contribution to the E field of interest.

Replace  $\infty$  with 500 and do a comparison of magnitudes.

$$|\vec{E}_y| \approx \frac{\sqrt{\mu_0 \epsilon_0} I}{2\pi \epsilon_0} \beta \left[ \sinh^{-1} \left[ \frac{500}{x} \right] - 1 \right]$$

$$\frac{|\vec{E}_x|}{|\vec{E}_y|} = \frac{\frac{1}{x}}{\beta \left[ \sinh^{-1} \left[ \frac{500}{x} \right] - 1 \right]}$$

will require a numerical comparison to get the picture

# Transmission Line Electric Fields (cont.)

comparing non-linear AC charge distribution

wrapping up

$$\frac{|\vec{E}_x|}{|\vec{E}_y|} = \frac{\frac{1}{x}}{\beta \left[ \sinh^{-1} \left[ \frac{500}{x} \right] - 1 \right]}$$

$$\beta = \frac{2\pi}{\lambda}$$

$$\frac{|\vec{E}_x|}{|\vec{E}_y|} = \frac{\lambda}{2\pi x \left[ \sinh^{-1} \left[ \frac{500}{x} \right] - 1 \right]}$$

$\lambda = 5000\text{km}$  (for  $f = 60\text{Hz}$ )

$$\frac{|\vec{E}_x|}{|\vec{E}_y|} = \frac{2500k}{\pi x \left[ \sinh^{-1} \left[ \frac{500}{x} \right] - 1 \right]}$$

wavelength (m) = 5000000  
integrated over (m) = 500

x	Ex	Ey	Ex / Ey
0.01	100.00	14.5E-6	6911999
0.02	50.00	13.6E-6	3677394
0.05	20.00	12.4E-6	1607043
0.10	10.00	11.6E-6	863983
0.20	5.00	10.7E-6	467136
0.50	2.00	9.6E-6	209362
1.00	1.00	8.7E-6	115167
2.00	0.50	7.8E-6	63983
5.00	0.20	6.7E-6	29982
10.00	0.10	5.8E-6	17206
20.00	0.05	5.0E-6	10069
50.00	0.02	3.9E-6	5145

It appears the magnitude of the radial or x component of the E field could be much greater than the magnitude of the parallel or y component (for any given line position of interest)

But lets look at a range of radial or x positions... from very close to the line to very far from the line. (since inverse hyperbolic sine is somewhat obscure)

## Conclusion

Using a constant linear charge distribution on the line is acceptable for any snapshot in time.

$$\sqrt{\frac{\mu_0}{\epsilon_0}} = \text{impdance of freespace} = 120\pi$$

$$\vec{E} = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{I \cos \omega t}{2\pi \epsilon_0 r} \hat{r}$$

this derivation is for one line with sinusoidal AC current and does not consider an "image" below a conducting earth surface



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