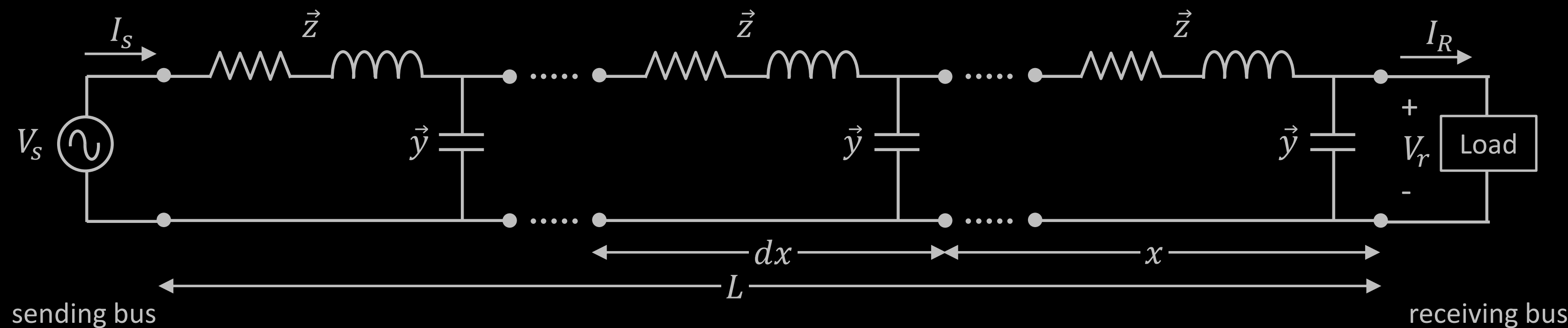
Long Transmission Line π Model

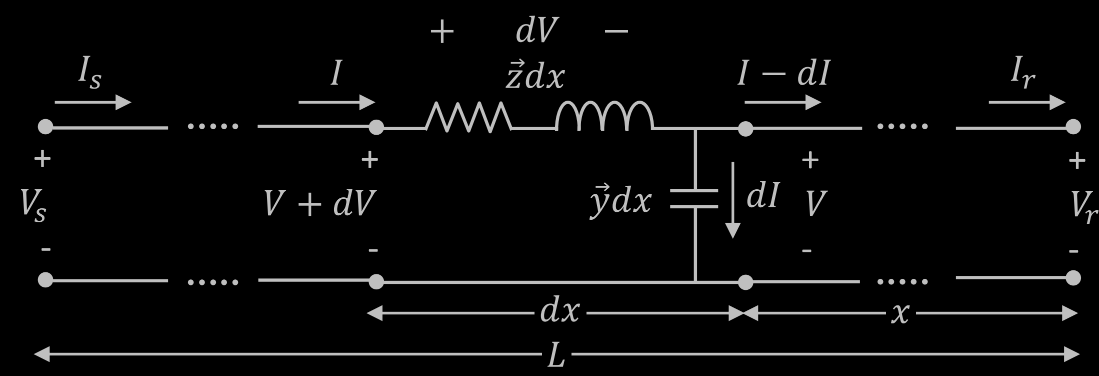
The long transmission line model is an exact model with no approximations:
the short and medium T-Line models are approximated from this model.
It uses "distributed" per-unit length line parameters:

$$\vec{z} = \frac{\vec{Z}_{total}}{length} = \frac{R_{total} + jX_{total}}{length}$$

$$\vec{y} = \frac{\vec{Y}_{total}}{length} = \frac{G_{total} + jB_{total}}{length} = \frac{jB_{total}}{length} \quad \textit{(ignoring shunt conductance)}$$



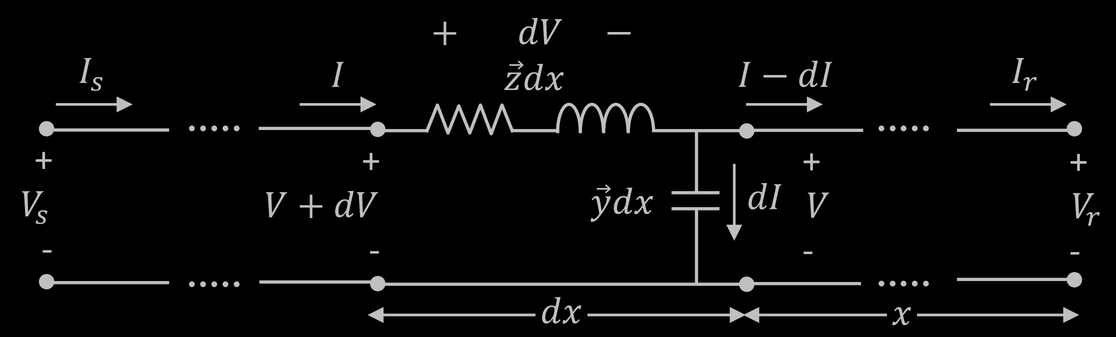
Long Transmission Line π Model (cont.)



consider a long transmission line with sending end voltage and current V_s *and* I_s.
where the receiving end voltage and current is V_r *and* I_r.
now consider an element of infinitely small length dx at a distance x from the receiving end.

$V+dV$ = *voltage on the sending side of the infinitely small line length dx*
I = *current entering the infinitely small line length dx*
dV = *voltage drop across the dx length*
V = *voltage on the receiving side of the dx length*
dI = *current entering the shunt admittance of the dx length*
$I - dI$ = *current leaving the infinitely small line length dx*
$\vec{z}\,dx$ = *the series impedance of this line length dx*
$\vec{y}\,dx$ = *the shunt admittance of this line length dx*
L = *total line length*
$\vec{Z} = \vec{z}L$ = *total line series impedance*
$\vec{Y} = \vec{y}L$ = *total line shunt admittance*

Long Transmission Line π Model (cont.)



voltage drop across length dx:

$$dV = I\vec{z}dx$$
$$\frac{dV}{dx} = I\vec{z}$$

current entering shunt admittance :

$$dI = V\vec{y}dx$$
$$\frac{dI}{dx} = V\vec{y}$$

differentiating both sides with respect to x:

$$\frac{d^2V}{dx^2} = \vec{z}\frac{dI}{dx}$$

$$\frac{d^2V}{dx^2} = \vec{z}\vec{y}V$$
$$\frac{d^2V}{dx^2} - \vec{z}\vec{y}V = 0$$

the solution to this second order differential equation has the form:

$$V = Ae^{x\sqrt{\vec{y}\vec{z}}} + Be^{-x\sqrt{\vec{y}\vec{z}}}$$

Long Transmission Line π Model (cont.)

$$V = Ae^{x\sqrt{\vec{y}\vec{z}}} + Be^{-x\sqrt{\vec{y}\vec{z}}}$$

differentiating with respect to x:

$$\frac{dV}{dx} = \sqrt{\vec{y}\vec{z}}Ae^{x\sqrt{\vec{y}\vec{z}}} - \sqrt{\vec{y}\vec{z}}Be^{-x\sqrt{\vec{y}\vec{z}}}$$

since:

$$\frac{dV}{dx} = I\vec{z}$$

then:

$$I\vec{z} = \sqrt{\vec{y}\vec{z}}Ae^{x\sqrt{\vec{y}\vec{z}}} - \sqrt{\vec{y}\vec{z}}Be^{-x\sqrt{\vec{y}\vec{z}}}$$

$$I = \frac{\sqrt{\vec{y}\vec{z}}Ae^{x\sqrt{\vec{y}\vec{z}}}}{\vec{z}} - \frac{\sqrt{\vec{y}\vec{z}}Be^{-x\sqrt{\vec{y}\vec{z}}}}{\vec{z}}$$

$$I = \frac{Ae^{x\sqrt{\vec{y}\vec{z}}}}{\sqrt{\frac{\vec{z}}{\vec{y}}}} - \frac{Be^{-x\sqrt{\vec{y}\vec{z}}}}{\sqrt{\frac{\vec{z}}{y}}}$$

next define …

Characteristic Impedance $\vec{Z}_o$

$$\vec{Z}_o = \sqrt{\frac{\vec{z}}{\vec{y}}} = \sqrt{\frac{z}{y}}\, e^{j\left(\frac{\theta_z}{2}-\frac{\pi}{4}\right)}$$

$$\theta_z = \tan^{-1}\left[\frac{x_L}{r}\right]$$

Propagation Constant $\vec{\gamma} = \alpha + j\beta$

$$\vec{\gamma} = \sqrt{\vec{y}\vec{z}} = \sqrt{yz}e^{j\left(\frac{\theta_z}{2}+\frac{\pi}{4}\right)}$$

$$\alpha = \sqrt{yz}\cos\left[\frac{\theta_z}{2}+\frac{\pi}{4}\right] = attenuation\ constant$$

$$\beta = \sqrt{yz}\sin\left[\frac{\theta_z}{2}+\frac{\pi}{4}\right] = phase\ constant$$

the propagation constant of a sinusoidal electromagnetic wave is a measure of the change undergone by the amplitude and phase of the wave as it propagates in a given direction

Long Transmission Line π Model (cont.)

$$\vec{Z}_o = \sqrt{\frac{\vec{z}}{\vec{y}}} \qquad \vec{\gamma} = \sqrt{\vec{y}\vec{z}}$$

$$I = \frac{Ae^{x\sqrt{\vec{y}\vec{z}}}}{\sqrt{\frac{\vec{z}}{\vec{y}}}} - \frac{Be^{-x\sqrt{\vec{y}\vec{z}}}}{\sqrt{\frac{\vec{z}}{\vec{y}}}} \qquad V = Ae^{x\sqrt{\vec{y}\vec{z}}} + Be^{-x\sqrt{\vec{y}\vec{z}}}$$

$$V = Ae^{\vec{\gamma}x} + Be^{-\vec{\gamma}x}$$

$$I = \frac{Ae^{\vec{\gamma}x}}{\vec{Z}_o} - \frac{Be^{-\vec{\gamma}x}}{\vec{Z}_o}$$

at $x = 0$:

$V = V_r$

$I = I_r + \cancel{dI}^{\,0}$ *ignore last admittance contribution*

$I = I_r$

$$I_r = \frac{A}{\vec{Z}_o} - \frac{B}{\vec{Z}_o} \qquad V_r = A + B$$

solving for A and B:

$$A = \frac{V_r + I_r\vec{Z}_o}{2} \qquad B = \frac{V_r - I_r\vec{Z}_o}{2}$$

rewriting with A and B substituted:

$$I = \frac{[V_r + I_r\vec{Z}_o]e^{\vec{\gamma}x}}{2\vec{Z}_o} - \frac{[V_r - I_r\vec{Z}_o]e^{-\vec{\gamma}x}}{2\vec{Z}_o}$$

$$I = \frac{V_r e^{\vec{\gamma}x}}{2\vec{Z}_o} + \frac{I_r\vec{Z}_o e^{\vec{\gamma}x}}{2\vec{Z}_o} - \frac{V_r e^{-\vec{\gamma}x}}{2\vec{Z}_o} + \frac{I_r\vec{Z}_o e^{-\vec{\gamma}x}}{2\vec{Z}_o}$$

$$I = \frac{V_r}{\vec{Z}_o}\underbrace{\left[\frac{e^{\vec{\gamma}x} - e^{-\vec{\gamma}x}}{2}\right]}_{\sinh\vec{\gamma}x} + I_r\underbrace{\left[\frac{e^{\vec{\gamma}x} + e^{-\vec{\gamma}x}}{2}\right]}_{\cosh\vec{\gamma}x}$$

$$V = \frac{[V_r + I_r\vec{Z}_o]e^{\vec{\gamma}x}}{2} + \frac{[V_r - I_r\vec{Z}_o]e^{-\vec{\gamma}x}}{2}$$

$$V = V_r\underbrace{\left[\frac{e^{\vec{\gamma}x} + e^{-\vec{\gamma}x}}{2}\right]}_{\cosh\vec{\gamma}x} + I_r\vec{Z}_o\underbrace{\left[\frac{e^{\vec{\gamma}x} - e^{-\vec{\gamma}x}}{2}\right]}_{\sinh\vec{\gamma}x}$$

Long Transmission Line π Model (cont.)

$$I = I_r \cosh \vec{\gamma} x + \frac{V_r}{\vec{Z}_o} \sinh \vec{\gamma} x \qquad V = V_r \cosh \vec{\gamma} x + I_r \vec{Z}_o \sinh \vec{\gamma} x$$

at $x = L$ *:*

$V = V_s$

$I = I_s$

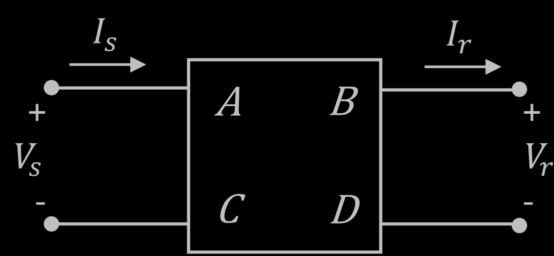
$$\boxed{I_s = I_r \cosh \vec{\gamma} L + \frac{V_r}{\vec{Z}_o} \sinh \vec{\gamma} L} \qquad \boxed{V_s = V_r \cosh \vec{\gamma} L + I_r \vec{Z}_o \sinh \vec{\gamma} L}$$

in matrix form:

$$\begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} \cosh \vec{\gamma} L & \vec{Z}_o \sinh \vec{\gamma} L \\ \dfrac{\sinh \vec{\gamma} L}{\vec{Z}_o} & \cosh \vec{\gamma} x \end{bmatrix} \begin{bmatrix} V_r \\ I_r \end{bmatrix}$$

expressed as two-port network:

$$\begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_r \\ I_r \end{bmatrix}$$



$$\therefore$$

$$A = D = \cosh \vec{\gamma} L$$

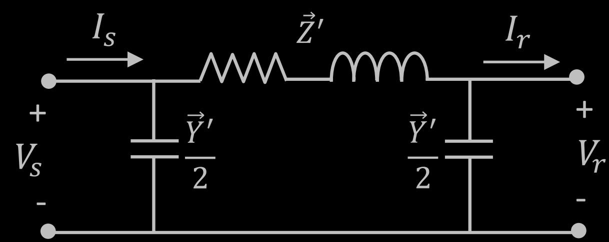
$$B = \vec{Z}_o \sinh \vec{\gamma} L$$

$$C = \frac{\sinh \vec{\gamma} L}{\vec{Z}_o}$$

Long Transmission Line π Model (cont.)

from two-port theory for π model:

thanks to International Journal of Advance Engineering Research and Technology (IJAERT)



$$\begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{\vec{Y}'}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & \vec{Z}' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\vec{Y}'}{2} & 1 \end{bmatrix} \begin{bmatrix} V_r \\ I_r \end{bmatrix}$$

$$\begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} 1 & \vec{Z}' \\ \frac{\vec{Y}'}{2} & 1 + \frac{\vec{Y}'\vec{Z}'}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\vec{Y}'}{2} & 1 \end{bmatrix} \begin{bmatrix} V_r \\ I_r \end{bmatrix}$$

$$\begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} 1 + \frac{\vec{Y}'\vec{Z}'}{2} & \vec{Z}' \\ \vec{Y}' + \frac{\vec{Y}'^2\vec{Z}'}{4} & 1 + \frac{\vec{Y}'\vec{Z}'}{2} \end{bmatrix} \begin{bmatrix} V_r \\ I_r \end{bmatrix}$$

$\therefore$

$$A = D = 1 + \frac{\vec{Y}'\vec{Z}'}{2}$$

$$B = \vec{Z}'$$

$$C = \vec{Y}'\left[1 + \frac{\vec{Y}'\vec{Z}'}{4}\right]$$

Long Transmission Line π Model (cont.)

from two-port theory:

$$A = D = 1 + \frac{\vec{Y}'\vec{Z}'}{2}$$

$$B = \vec{Z}'$$

$$C = \vec{Y}'\left[1 + \frac{\vec{Y}'\vec{Z}'}{4}\right]$$

from previous long T-Line derivation:

$$A = D = \cosh \vec{\gamma}L$$

$$B = \vec{Z}_o \sinh \vec{\gamma}L$$

$$C = \frac{\sinh \vec{\gamma}L}{\vec{Z}_o}$$

∴

3 equations
2 unknowns
(Z' and Y')

$$\left\{\begin{aligned} & 1 + \frac{\vec{Y}'\vec{Z}'}{2} = \cosh \vec{\gamma}L \\ & \vec{Z}' = \vec{Z}_o \sinh \vec{\gamma}L = \sqrt{\frac{\vec{z}}{\vec{y}}}\sinh \vec{\gamma}L \\ & \underbrace{\vec{Y}'\left[1 + \frac{\vec{Y}'\vec{Z}'}{4}\right] = \frac{\sinh \vec{\gamma}L}{\vec{Z}_o} = \sqrt{\frac{\vec{y}}{\vec{z}}}\sinh \vec{\gamma}L}_{\text{avoid this second order equation when solving}} \end{aligned}\right.$$

now solve for $\vec{Z}'$ and $\vec{Y}'$:

Long Transmission Line π Model (cont.)

solve for $\vec{Z}'$

$$\vec{Z}' = \sqrt{\frac{\vec{z}}{\vec{y}}}\sinh\vec{\gamma}L$$

$$\vec{Z}' = \frac{\sqrt{\vec{z}}}{\sqrt{\vec{y}}}\sinh\vec{\gamma}L$$

$$\vec{Z}' = \frac{\sqrt{\vec{z}}\vec{\gamma}L}{\sqrt{\vec{y}}}\frac{\sinh\vec{\gamma}L}{\vec{\gamma}L}$$

$$\vec{Z}' = \frac{\sqrt{\vec{z}}\sqrt{\vec{y}\vec{z}}L}{\sqrt{\vec{y}}}\frac{\sinh\vec{\gamma}L}{\vec{\gamma}L}$$

$$\vec{Z}' = \frac{\vec{z}L\sqrt{\vec{y}}}{\sqrt{\vec{y}}}\frac{\sinh\vec{\gamma}L}{\vec{\gamma}L}$$

$$\vec{Z}' = \vec{z}L\frac{\sinh\vec{\gamma}L}{\vec{\gamma}L}$$

$$\boxed{\vec{Z}' = \vec{Z}\frac{\sinh\vec{\gamma}L}{\vec{\gamma}L}}$$

solve for $\vec{Y}'$ given $\vec{Z}'$

$$1 + \frac{\vec{Y}'\vec{Z}'}{2} = \cosh\vec{\gamma}L$$

$$1 + \vec{z}L\frac{\sinh\vec{\gamma}L}{\vec{\gamma}L}\frac{Y'}{2} = \cosh\vec{\gamma}L$$

$$\vec{Y}' = \frac{2\vec{\gamma}L}{\vec{z}L}\frac{\cosh\vec{\gamma}L - 1}{\sinh\vec{\gamma}L}$$

$$\vec{Y}' = \frac{2\vec{\gamma}}{\vec{z}}\tanh\frac{\vec{\gamma}L}{2}$$

$$\vec{Y}' = \frac{2\vec{\gamma}\vec{\gamma}L}{2\vec{z}}\frac{\tanh\frac{\vec{\gamma}L}{2}}{\frac{\vec{\gamma}L}{2}}$$

$$\vec{Y}' = \frac{\vec{y}\vec{z}L}{\vec{z}}\frac{\tanh\frac{\vec{\gamma}L}{2}}{\frac{\vec{\gamma}L}{2}}$$

$$\vec{Y}' = \vec{y}L\frac{\tanh\frac{\vec{\gamma}L}{2}}{\frac{\vec{\gamma}L}{2}}$$

$$\boxed{\vec{Y}' = \vec{Y}\frac{\tanh\frac{\gamma L}{2}}{\frac{\vec{\gamma}L}{2}}}$$

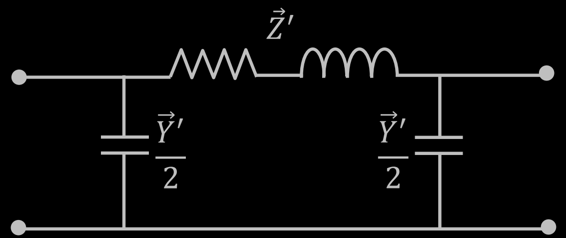
Long Transmission Line π Model (cont.)

we now have the long transmission line π model
where the lumped impedance and admittance can be "corrected"

$$\boxed{\vec{Z}' = \vec{Z}\,\frac{\sinh \vec{\gamma}L}{\vec{\gamma}L}}$$

$$\boxed{\vec{\gamma} = \sqrt{\vec{y}\vec{z}}}$$

where $\vec{y}$ and $\vec{z}$ are distributed and complex quantities



long transmission line π model

$$\boxed{\vec{Y}' = \vec{Y}\,\frac{\tanh \frac{\gamma L}{2}}{\frac{\vec{\gamma}L}{2}}}$$

important to note…
the long line shunt conductance is not zero (but usually ignored).

$$\vec{Y}' = G' + jB'$$

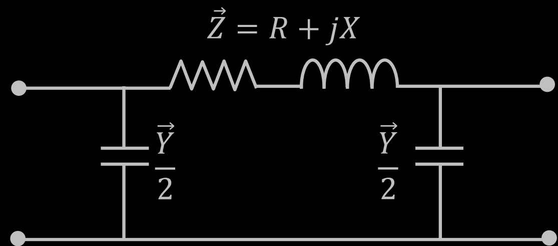
for lines that are not considered "long" … γL is small
and the long line model reduces to the simpler "medium" line model

using small angle approximations for hyperbolic functions:

$$\sinh \vec{\gamma}L \approx \vec{\gamma}L \qquad \tanh \frac{\vec{\gamma}L}{2} \approx \frac{\vec{\gamma}L}{2}$$

$$\therefore$$

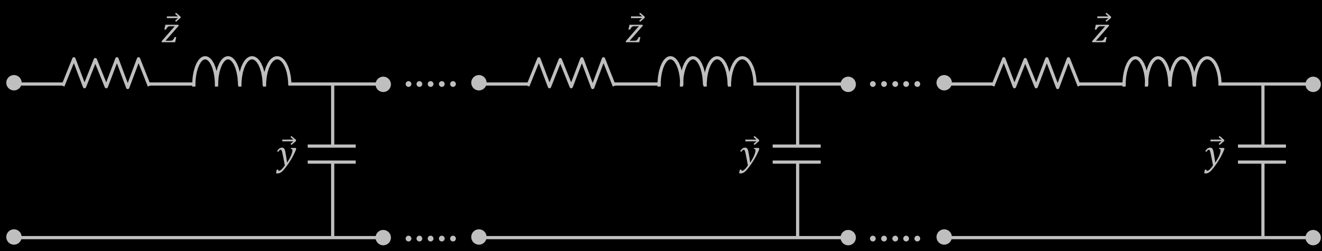
$$\vec{Z}' = \vec{Z}\,\frac{\sinh \vec{\gamma}L}{\vec{\gamma}L} \approx \vec{Z}\,\frac{\vec{\gamma}L}{\vec{\gamma}L} = \vec{Z}$$



medium transmission line π model

Long Transmission Line π Model (cont.)

(if line could be considered lossless)



$$\boxed{\vec{Z}' = \vec{Z}\frac{\sinh \vec{\gamma} L}{\vec{\gamma} L}}$$

$$\boxed{\vec{Y}' = \vec{Y}\frac{\tanh \frac{\gamma L}{2}}{\frac{\vec{\gamma} L}{2}}}$$

consider the series impedance

$\vec{z} = r + jx$

if the line could be considered lossless…

$\vec{z} \approx jx$

ignoring shunt conductance…

$\vec{y} \approx jb$

$\therefore$

$\vec{\gamma} \approx \sqrt{jbjx} = j\sqrt{bx}$

the propagation constant becomes purely imaginary.

$$\sinh \phi = \frac{e^{\phi} - e^{-\phi}}{2} \qquad \tanh \phi = \frac{e^{\phi} - e^{-\phi}}{e^{\phi} + e^{-\phi}}$$

$$\sin \phi = \frac{e^{j\phi} - e^{-j\phi}}{2j} \qquad \tan \phi = \frac{e^{j\phi} - e^{-j\phi}}{j[e^{j\phi} + e^{-j\phi}]}$$

recall

substituting the purely imaginary propagation constant:

$$\frac{\vec{Z}'}{\vec{Z}} = \frac{\sinh j\sqrt{bx}L}{j\sqrt{bx}L}$$

$$\frac{\vec{Z}'}{\vec{Z}} = \frac{e^{j\sqrt{bx}L} - e^{-j\sqrt{bx}L}}{2j\sqrt{bx}L}$$

$$\frac{\vec{Z}'}{\vec{Z}} = \frac{\sin \sqrt{bx}L}{\sqrt{bx}L}$$

$$\frac{\vec{Z}'}{\vec{Z}} = \frac{\sin \sqrt{bLxL}}{\sqrt{bLxL}}$$

$$\boxed{\frac{\vec{Z}'}{\vec{Z}} = \frac{\sin \sqrt{BX}}{\sqrt{BX}}}$$

$$\frac{\vec{Y}'}{\vec{Y}} = \vec{Y}\frac{\tanh \frac{j\sqrt{bx}L}{2}}{\frac{j\sqrt{bx}L}{2}}$$

$$\frac{\vec{Y}'}{\vec{Y}} = \frac{e^{j\frac{\sqrt{bx}L}{2}} - e^{-j\frac{\sqrt{bx}L}{2}}}{j\frac{\sqrt{bx}L}{2}\left[e^{j\frac{\sqrt{bx}L}{2}} - e^{-j\frac{\sqrt{bx}L}{2}}\right]}$$

$$\frac{\vec{Y}'}{\vec{Y}} = \frac{\tan \frac{\sqrt{bx}L}{2}}{\frac{\sqrt{bx}L}{2}}$$

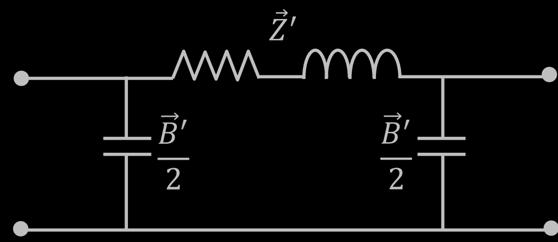
$$\boxed{\frac{\vec{Y}'}{\vec{Y}} = \frac{\tan \frac{\sqrt{BX}}{2}}{\frac{\sqrt{BX}}{2}}}$$

the correction factors are now purely real

Long Transmission Line π Model (cont.)

(if line could be considered lossless)

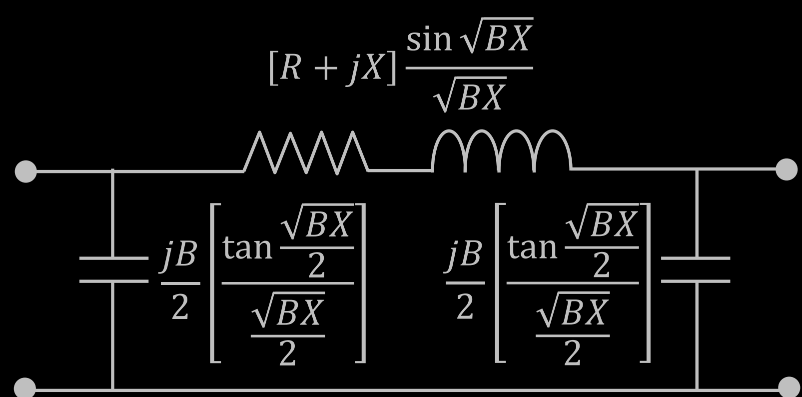
$$\boxed{\vec{Z}' = \vec{Z}\,\frac{\sin\sqrt{BX}}{\sqrt{BX}}}$$



$$\boxed{\frac{\vec{B}'}{2} = \frac{\vec{B}}{2}\,\frac{\tan\frac{\sqrt{BX}}{2}}{\frac{\sqrt{BX}}{2}}}$$

long transmission line π model

(with lossless propagation factor)



Long Transmission Line π Model (cont.)

how long does a transmission line need to be for it to be considered long?

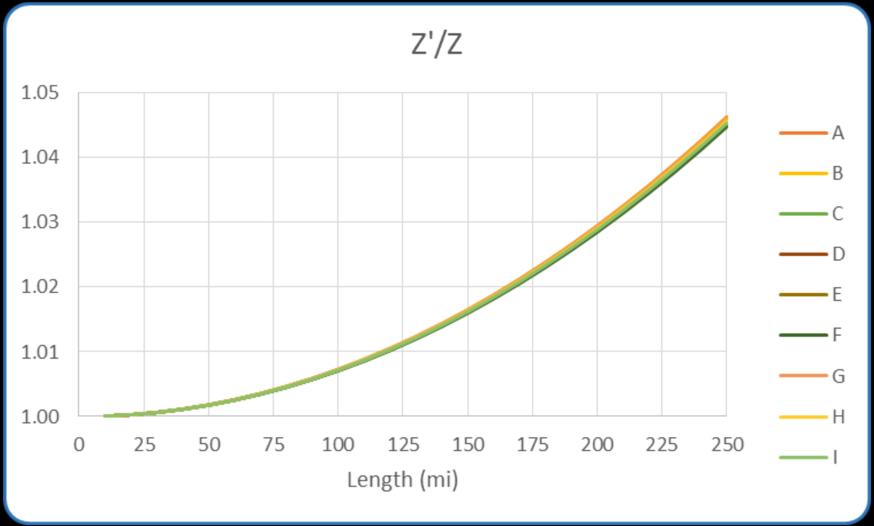
lets compare the long line "corrected" impedance and admittance to the "non-corrected" impedance and admittance for a few conductors and various voltages

$$\frac{\vec{Z}'}{\vec{Z}} = \frac{\sinh \gamma L}{\gamma L}$$

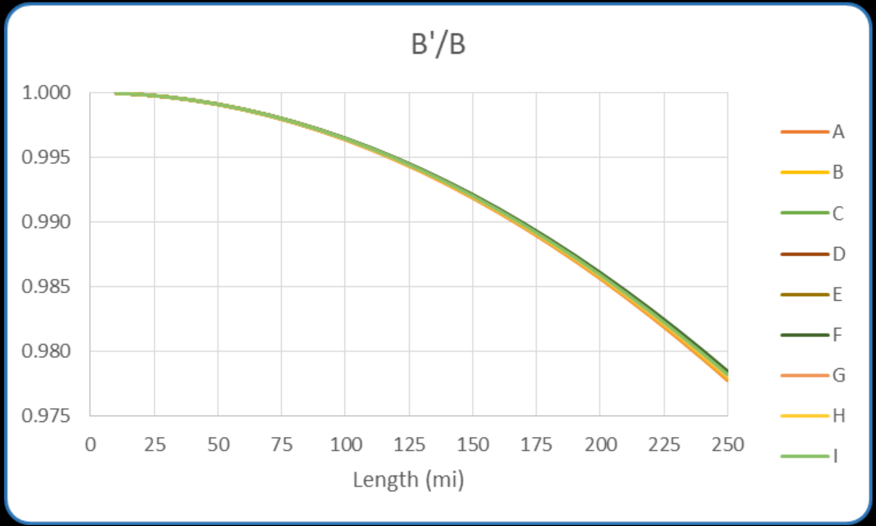
conductor	kV	R(pu/mi)	X(pu/mi)	B (pu/mi)	
795 ACSR	69	0.002917	0.013888	0.000309	A
795 ACSR	138	0.000729	0.003702	0.001156	B
795 ACSR	345	0.000117	0.000658	0.006474	C
2x795 ACSR	69	0.001458	0.009133	0.000465	D
2x795 ACSR	138	0.000365	0.002513	0.001683	E
2x795 ACSR	345	0.000058	0.000468	0.008999	F
1272 ACSR	69	0.001896	0.013392	0.000321	G
1272 ACSR	138	0.000474	0.003578	0.001198	H
1272 ACSR	345	0.000076	0.000638	0.006684	I

$$\frac{\vec{Y}'}{\vec{Y}} = \frac{\tanh \frac{\gamma L}{2}}{\frac{\gamma L}{2}}$$

corrected values are effectively independent of conductor and voltage



corrected impedance goes up with line length
has more change than admittance
1% error at 120 miles
2% error at 165 miles



corrected admittance goes down with line length
has less change than impedance
1% error at 160 miles
2% error at 235 miles

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